THE EFFECT OF TAXATION ON THE VALUATION OF OPTIONS

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(INTRODUCTION)

Investors have long been using "return on investment" criteria for investment decisions related to buying and writing options. However, this approach neglects the variability of return, i.e. the riskiness of the investment. Recently Black and Scholes (1) derived a general equilibrium call pricing model. They demonstrated that it was possible to form a riskless portfolio containing a stock and its call options. Based on this portfolio, they derived a formula for the value of an option enabling an investment decision to be made by comparing market prices with theoretical prices. Black and Scholes tested their model and the efficiency of American option traders in establishing option prices, by comparing the theoretical and market prices. (2) In their model, they have neglected the effect of taxation. It is shown here that the inclusion of taxation significantly affects theoretical prices.

The Black and Scholes Model

The importance of the Black and Scholes Call Pricing Model is that it provides the means of calculating the fair value of an option. Based on that value, the investor can decide whether he should buy or sell that option in order to improve his investment performance. Return parameters can then be calculated based on optimal strategy.

The Black and Scholes model is based on the fact that it is possible to have a hedge long in the underlying stock and short in its option in such a way, that any profit resulting from an instantaneous increase in stock price is offset exactly by a loss on the option position, and vice versa, for small changes in the stock price. Thus, if the quantities of the stock and its option in the hedge are continuously adjusted in the appropriate manner as the stock price changes, then the return on the hedge becomes riskless. Since, market risk is eliminated from the hedge, it must earn the riskless rate of return in equilibrium.

The price of the option, as calculated by Black and Scholes, is given in Appendix A. It is a function of the market price of the stock, the striking price of the option, the duration of the option (all known quantities), the risk-free short term rate of interest, (an observable quantity), and the variance rate of the return on the stock. It is worth noting that the option price does not depend on the expected return on the underlying security. As it is suggested that the value given by the above model be used as criterion for option investment decisions it is mandatory to ensure that correct values are substituted in the valuation formula. Any uncertainty is associated only with the values of the interest rate and the variance rate of return.

The total variance of the return on the common stock is often fairly constant, and hence can be estimated from time series data. Since the formula is a function of the instantaneous variance of the return it should be estimated by using the daily returns of the underlying security. Ideally, of course, the variance of the return of all transactions should be used but this method is not practicable. Unfortunately, one can never be certain that one is using the correct variance rate. Any analysis based on the historical variance rate of the return assumes that that variance rate will remain the same in the future. Therefore, it is worth noting that this may not be true over longer periods of time. Thus, the volatility of stock prices in Australia has varied considerably during the last few years. During 1974 and 1975 most stocks have shown much higher volatilities than since the beginning of 1976. It is a reasonable approach, however, to attach more importance to more recent values of the volatilities.

Now that the valuation of options has been considered, the importance of options in portfolios will be discussed next.

Portfolio Effects of Options

The portfolio implications of a hedge consisting of an option and its underlying security are most important. It demonstrates that it is possible to eliminate market or systematic risk from holding a portfolio of stocks which have traded options, by writing options against those stocks. Thus, it is no longer necessary for a portfolio manager to reduce market risk by selling the high volatility stocks from his portfolio and investing in less volatile stocks or other low risk investments, but to simply write options against the high volatility stocks he holds. The practical implication of this will obviously increase as the number of traded options increases.

Theory tells us how many options to write against the stocks held to obtain a low risk hedge, from which virtually all market risk has been eliminated for small stock price changes. This ratio of the number of stocks to the number of options held is called the hedge ratio and its value is given in Appendix A. The hedge ratio is, there-
fore, also the ratio of the change in the option price to the change in the stock price, for small changes in the stock price.

The Effect of Taxation on the Valuation Formula

If our basic strategy is to base our option investment decisions on our valuation model, we must ensure that our theoretical values are reliable. The formulae given in Appendix A do not take into consideration the effect of taxation. It is shown here, that the value of an option is significantly altered by the marginal tax rate of the investor, especially if the time to maturity is long.

The derivation for the theoretical option price is given in Appendix B. The difference from the value given by the Black and Scholes formula is due to the fact, that an investor normally has to pay tax on interest earned. The formula remains valid for the institutional investor whose marginal tax rate is zero, but the formula simply reduces to that given in Appendix A.

It must be noted that the formula does not take into account dividend payments of the underlying stock. Thus, it can only be applied directly to value Woodside-Burmah options and other options whose underlying security does not pay a dividend during the life of the particular option. The problem arises with a stock that pays a dividend because there is a possibility that the option will be exercised prior to expiration. On the other hand, an American option that has no payouts (i.e. dividends) should never be exercised prior to expiration.

The value of an option whose underlying security is paying a dividend can be found by carrying out two calculations using the formula in Appendix B. In the first calculation, the present value of the dividends is subtracted from the stock price before the substitution is made. In the second calculation, the present value of all dividends except the last is subtracted from the stock price and the time to maturity is shortened as if the last ex-dividend date was the expiration date to be used in the valuation formula. The higher of the two values is taken as the option price. Valuing the option this way assumes that the option holder will maximise the return on his investment.

Implication of the Model

As an example of the use of the model, calculated Woodside-Burmah option prices, on 29 March 1977, (closing price $1.05), are shown in Table 1. The variance rate of return was calculated using adjusted daily closing prices over a six month period and was found to be 0.37 for Woodside-Burmah shares. The interest rate taken was the buying rate used by the Accepting Houses Association for prime non-bank commercial bills and having the same maturity dates as the options. This, although higher than the risk-free rate is more appropriate as has been shown by Merton. While the mathematical proof is lengthy and complicated, one of the main reasons is easy to understand. The hedge ratio is a function of all six option price parameters (see Appendix B), hence it continually varies. In order to earn only the risk-free rate the hedge must be continuously rebalanced by selling or buying options and/or shares. But, this implies an intolerably high level of commission cost. Hence, the hedge would not be continuously rebalanced in a practical situation, resulting in a slightly variable rate of return on the hedge.

As can be seen from Table 1, the 'fair value' of the option will differ significantly, depending on the tax status of the investor. The values in the table were not rounded to the nearest cent to show more clearly the effect of the inclusion of taxation. Moreover, if the calculated prices are used as a guide for making 'bids', rounding causes the loss of useful information. A trading strategy could be to buy 'underpriced' options and sell 'overpriced' ones. An 'under-priced' option is meant to be an option that is trading at a price under the calculated value. It can be noted, for instance, that in Table 1, the June 1.25 options are 'fairly' priced and the June 1.00 options are 'underpriced'. Also, the Dec 1.00 options are 'underpriced' while the Dec 1.25 options are 'overpriced', except for the tax exempt investor. These observations are not unusual, since options that are way out of the money tend to be 'overpriced' and options that are way into the money tend to be 'underpriced'.

The higher the marginal tax rate of the investor, the lower is his calculated option value. This can be seen from the expression derived in Appendix B, which gives the change in option value resulting from a change in taxation rate, for small changes in taxation rate. It can also be observed that the effect of taxation is largest for the nine months options (about 11% for Dec 1.25 option) and it is less for the six and three months options. In fact, it decreases as the life of the option decreases even for the same option. This can be seen from Fig. 1, which shows the time value premium as a function of the time to expiration date for the investor who pays no tax and the investor whose marginal tax rate is 0.65.

Table 2 gives the hedge ratios corresponding to Table 1. Thus for June 1.00 options the hedge ratio is about 2/s (more precisely, it is in the range of 0.64 to 0.66). Then the low risk hedge is the one which is short three options and long two shares. If the share price increases 3 cents then the option price increases 2 cents and vice versa. An investor holding the above hedge loses 6 cents on his options and gains 6 cents on his share when the share price increases 3 cents. Hence, he is isolated from small changes in stock prices.
Conclusion

In this paper option values are calculated taking into account the effect of taxation payable on interest. It is shown that this results in lower option values and hedge ratios and its effect decreases as the life of the option decreases.

FOOTNOTES

1 Black and Scholes investigated European options and included transaction costs in their calculations. Their pricing model, however, can easily be extended for the valuation of American options (3). Options which may only be exercised on the expiration date are generally referred to as European options. Exchange Traded Options, which may be exercised before the expiration date are generally referred to as American options.

2 The variance rate of return on a stock is the limit, as the size of the interval of measurement goes to zero, of the variance of the return over that interval dividend by the length of the interval.

3 The option premium is equal to the sum of the intrinsic value and the time value premium.

4 $S = 1.05$, $X = 1.00$ were used as parameters. $S$ was assumed to remain unchanged, but adjusted values were used for $r$ to take into account the various times to expiration date.

BIBLIOGRAPHY


APPENDIX A

The price of an option as calculated by Black and Scholes is given by [1]:

\[ P = SN(a) - Xe^{-rT}N(b) \]
\[ = \frac{\ln(S/X) + (r + \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \]
\[ a = \frac{\ln(S/X) + (r + \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \]
\[ b = a - \sigma \sqrt{T} \]

where

- $P$ = the price of an option for a single share of stock
- $S$ = the current market price of the stock
- $X$ = the exercise price of striking price of the option
- $r$ = the 'risk-free' short term rate of interest
- $T$ = the duration of the option
- $\sigma^2$ = the variance rate of the return on the stock
- $N(a, N(b))$ = the value of the cumulative normal density function.

The hedge ratio, $H$, the number of shares to balance against each option for a riskless hedge, is given by (1):

\[ H = N(a) \]
APPENDIX B

Following the method of solution used by Black and Scholes (1) let us consider a hedged position one stock long and 1/P\textsubscript{S} options short. The value of the equity in the position is:

\[ S - P/P\textsubscript{S} \]

where all the variables are as defined in Appendix A. Let x be the time variable. Then using stochastic calculus, a small change in option price, \( \Delta P \), can be written as

\[ \Delta P = P\textsubscript{S} \Delta S + \frac{1}{2} P\textsubscript{SS} v^2 S^2 \Delta x + P\textsubscript{X} \Delta x \]

where \( P\textsubscript{X} \) is the partial derivative of \( P \) with respect to \( x \).

The change in equity in the hedge must equal the after tax return earned by risk free interest rate. Therefore

\[ -(\frac{1}{2} P\textsubscript{SS} v^2 S^2 + P\textsubscript{X}) \Delta x / P\textsubscript{S} = (S - P/P\textsubscript{S}) (1 - t) r \Delta x \]

where \( t \) is the taxation rate applicable to interest payments.

Putting \( R = (1 - t) r \)

and rearranging, we have a differential equation for the value of the option

\[ P\textsubscript{X} = R P - R S P\textsubscript{S} - \frac{1}{2} v^2 x^2 P\textsubscript{SS} \]

This is the differential equation solved by Black and Scholes (1) subject to the boundary condition

\[ P(S,T) = S - X \quad S \geq X \]
\[ = 0 \quad S < X \]

and the solution is given in Appendix A.

Substituting back for \( R \), we get the value of option as

\[ P = S N(c) - X e^{-(1 - t) r T} N(d) \]
\[ c = \frac{\ln(S/X) + [(1 - t) r + \frac{1}{2} v^2] T}{v \sqrt{T}} \]
\[ d = c - v \sqrt{T} \]

The hedge ratio is given by

\[ H = N(c) \]

The change in option price due to small changes in taxation rate, \( \Delta t \), can be obtained by partially differentiating the option price with respect to the taxation rate. After rearranging and simplifying we get

\[ \Delta P = -r T X e^{-r(1 - t) T} N(d) \Delta t \]

Thus increasing the taxation rate decreases the option price.
TABLE 1: CALCULATED OPTION PRICES
(IN CENTS)
OF WOODSIDE-BURMAH SHARES

<table>
<thead>
<tr>
<th>Option Issue</th>
<th>Marginal Tax Rate 0.45</th>
<th>Marginal Tax Rate 0.55</th>
<th>Marginal Tax Rate 0.65</th>
<th>Actual Last Sale Price</th>
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<td>15.6</td>
<td>15.4</td>
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<td>2.4</td>
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<td>Sept 1.00</td>
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<td>21.5</td>
<td>21.2</td>
<td>20.9</td>
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<td>12.2</td>
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TABLE 2: CALCULATED HEDGE RATIOS FOR
WOODSIDE-BURMAH SHARES

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*Fig. 1 Time value premium as a function of time to expiration date*