A newly developed model can help investors to calculate theoretical prices for fixed-interest securities. John Bellingham explains.

Deriving a zero curve is an attempt to model an interest-rate structure which explains the prices and related yields to maturity on securities observed in the marketplace. This technique can be used to price securities consistently and to identify anomalies caused by supply-and-demand pressures.

For example, a zero curve model can explain the difference in yield for two bonds with the same maturity date and issuer, but significantly different coupon rates.

Another example is to price pre-tender a new bond issue with a zero curve derived from the market yields on a group of actively-traded or "hot" stocks.

**THEORY**

For a group of securities with the same risk and liquidity characteristics, it is possible to derive a continuous function which represents the present value of any cash payment from that group of securities. Let us call this a discount curve.

The discount curve represents the present value of $1 on the Y axis against the term-to-payment on the X axis.

A zero curve can be deduced from the discount curve by a simple mathematical relationship.

**Properties of the discount curve**

- It appears only in the quadrant where X and Y are positive or zero. This means that the present value and the term-to-payment are never negative.
- The present value of $1 starts at 1 when the term to payment is 0.
- The present value of $1 decreases as the term to payment increases and approaches 0 as the time-to-payment approaches infinity.

If the rate of interest or discount is constant throughout time, then the discount function can be represented by an exponential curve:

\[ PV(t) = EXP(-r*t) \]

where

- \( t \) is term-to-payment
- \( r \) is the constant continuous rate of interest
- \( PV(t) \) is the present value of $1 in \( t \) years time.

In reality, most fixed-interest markets are more complex and \( r \) is not constant over time-to-payment. If \( EXP(-r*t) \) is replaced with a cubic polynomial function of \( EXP (-r*t) \) over a range of terms-to-payment, the discount function becomes:

![Discount Curve Example](image)
\[ PV(t) = a_0 \cdot \exp(-r_t) + a_1 \cdot \exp(-2r_t) + a_2 \cdot \exp(-3r_t) \]

where \( a_0, a_1, a_2 \) and \( a_3 \) are constant within each segment of the curve.

This function is suitable because it allows adjoining sections of the discount curve (separated by points known as knot points) to flow smoothly into each other.

**THE KNOT POINT**
- Two adjoining curves can have the same \( X \) and \( Y \) values at the intervening knot point.
- Two adjoining curves can have the same gradient or first derivative at the intervening knot point.
- Two adjoining curves can have the same curvature or second derivative at the intervening knot point.

A set of the constants \( a_0, a_1, a_2 \) and \( a_3 \) could be found for each segment of the discount curve.

Clearly \( a_0 \) is zero in the initial segment of the discount curve as \( PV(0) \) is equal to 1.

The process of fitting the segments of the discount curve to data is simplified by transforming the term-to-payment variable.

\[ \text{let } t = 1 - \exp\left( -a \cdot x \right) \quad \text{or, inversely,} \]
\[ \text{let } x = -\frac{\ln (1-t)}{a} \]

where \( a \) is chosen as a parameter, \( a > 0 \)

The equation for \( PV(t) \) reduces to:

\[ PV(t) = b_0 + b_1 \cdot x + b_2 \cdot x \cdot x + b_3 \cdot x \cdot x \cdot x \]

where \( b_0, b_1, b_2 \) and \( b_3 \) are constant within each segment of the curve.

Clearly \( b_0 \) is 1 when \( t = 0 \) in the initial segment of the discount curve, as \( x \) is also 0 if \( t = 0 \).

**ESTIMATION OF PARAMETERS**

The cubic polynomial in \( x \) is a convenient form for estimating the parameters \( b_0 \) to \( b_3 \) by linear regression.

For one particular security, the gross price is given as

\[ GP = \text{Sum} \left( PV(t) \cdot CF_t \right) \]

where \( CF_t \) is the cashflow at time \( t \)

If all the cashflows fall in the initial segment of the discount curve, the gross price calculated from the estimated zero curve can be compared with the market gross price.

The parameters can be optimised to minimise the difference of actual to fitted gross price over a range of securities. In practice, the differences are weighted by \( 1/ \text{duration} \) before optimising the parameters.
RESULTS

Government bonds
The CGS zero curve illustrated on page 20 was derived for a group of commonwealth government stocks using yields current at approximately 11am on 22 July 1994.

The derived parameters were for one curve segment only, covering terms-to-payment from 0 to 13 years.

Parameters
\begin{align*}
a &= 0.1174 \\
b_0 &= 1.0141 \\
b_1 &= -0.0669 \\
b_2 &= -0.5139 \\
b_3 &= 0.2371
\end{align*}

90-day bank bill futures
The bank bill zero curve example was derived for a strip of 90-day bank bill futures current at approximately 11am on 19 October 1994.

The derived parameters were for one curve segment only, covering terms-to-payment from 0 to 3 years.

Parameters
\begin{align*}
a &= 0.226599 \\
b_0 &= 1.0000 \\
b_1 &= -0.2566 \\
b_2 &= -0.5665 \\
b_3 &= 0.1854
\end{align*}

SUMMARY

It appears possible in practice to model the underlying term structure of interest rates using regression methods to fit cubic spline curve segments to make up a discount curve. The resultant zero curve derived from the discount curve can be used to calculate theoretical prices for fixed-interest securities for comparison with actual market prices.

This comparison has produced quite a close fit even with only a single segment of the curve being fitted.

Additional flexibility can be added, if required, by dividing the discount curve into segments and fitting parameters to each segment.

REFERENCE