Share price ratios, developed by Australian Stock Exchange Derivatives, are relative performance contracts devised to take advantage of differences between individual stock prices and the All-Ordinaries index.

NEVILLE HATHAWAY presents a pricing formula for this new derivative product.

Share price ratios, developed by ASX Derivatives and others, are simple contracts that will pay off on the relative performance of a stock price to the All-Ordinaries index (AOI). As an illustration, a BHP ratio price will be 1,000 times the BHP share price (in cents) divided by the AOI, and the contract will be $10.00 times the ratio price. If the BHP share price is $18.00 and the AOI is 2,000.0, then the ratio price is 900 and the contract will be $9,000.00. These contracts are discussed in depth in a report by Hathaway (1994) and only the basic essentials are discussed here in order to derive a pricing equation. The exact pricing equation derived here updates the heuristic pricing formulas presented in that report.

A ratio is a contract that (ignoring contract multipliers):

• pays off on the difference between the opening ratio of a stock price to an index price, $S_0/I_0$, and the closing ratio value, $S_t/I_t$;
• is a smooth function of time, stock price and index price; and
• is a derivative security over two traded securities, the stock and the index (which is equivalent to a portfolio of stocks).

In summary, a ratio contract entered into at time $t = 0$ is described by a function $R(S,I,t)$ and the maturity payoff function:

$$R(S,I,t=T) = \left[ \frac{S_T}{I_T} \right] - \left[ \frac{S_0}{I_0} \right]$$

### UNDERLYING SECURITY PRICING SET-UP

We employ the standard set-up for pricing any derivative contract (see, for example, Hull 1993). Suppose the stock price and the index both follow random walks, with some correlation between the stock and the index.

$$\frac{dS}{S} = \mu_S dt + \sigma_S dZ_1$$

$$\frac{dI}{I} = \mu_I dt + \sigma_I dZ_2$$

$$dZ_1 \times dZ_2 = \rho$$

With this (standard) set-up, the ratio contract, and all other such derivative contracts over traded securities, must satisfy the fundamental pricing equation (see, for example, Hull, Chapter 12)

$$\frac{1}{2} \sigma_S^2 S^2 R_{SS} + \frac{1}{2} \sigma_I^2 R_{II} + \rho \sigma_S \sigma_I R_{SI} + (r - d_i)SR_i + (r - d_s)IR_i + R_i - rR = 0$$

(1)

where $r$ is the interest rate and $d_i$ and $d_s$ are the dividend yields on the index and stock respectively (all of which are assumed constant). The pricing formula can be derived for discrete dividend payments but, just as for options, the solution will be numerical rather than analytic. All derivatives that are smooth functions of $S$, $I$, and $t$ must satisfy this equation. The only way such derivatives differ is in their payoff construction.

### TRANSFORMING THE EQUATION

Intuitively, the ratio contract should be a
function of the relative values of S and I, not the absolute values. To this end, introduce a new variable, X, and a new pricing function, G, defined by:

\[ X = \frac{S}{I} = S_0^{-1} \text{ and } R(S,I,t) = G(X,t) \]

Then, substituting these into the fundamental equation, we get

\[
\begin{align*}
\left[ 1 - \frac{1}{2}\sigma_r^2 + \frac{1}{2}\sigma_j^2 - \rho \sigma_r \sigma_j \right] G_x + \sigma_j^2 - \rho \sigma_j \sigma_r \right] X^2 G_x + \sigma_r^2 - \rho \sigma_r \sigma_r \right] XG_x + G_r - r G = 0
\end{align*}
\]

with the maturity condition

\[ G(X,t=T) = X_T - X_0 \]

The reduced equation (2a) is of the standard Black-Scholes form and the maturity payoff is analogous to a forward contract payoff. This suggests the solution is analogous to that of a forward contract. However, the parameters of the fundamental equation (2) are not the standard ones so the solution will not be a standard forward contract.

The solution to the ratio pricing equation (proof is available from the author on request) for a contract entered into at time t and due to mature at time T is

\[
R(S,I,t) = \left[ \frac{S_t}{I_t} \right] e^{-r(T-t)} \left[ 1 - \frac{1}{2}\sigma_r^2 + \frac{1}{2}\sigma_j^2 - \rho \sigma_r \sigma_j \right] \left[ XG_x + G_r - r G \right]
\]

This gives the fair value at time t for the expected contract payoff at time T. In most circumstances, investors would want the value of a newly entered contract, namely, the value at time t of a new contract that will expire at time T. This of course just means a re-focus of the zero point in (3), namely

\[
R(S,I,t) = \left[ \frac{S_t}{I_t} \right] e^{-r(T-t)} \left[ 1 - \frac{1}{2}\sigma_r^2 + \frac{1}{2}\sigma_j^2 - \rho \sigma_r \sigma_j \right] XG_x + G_r - r G = 0
\]

with the maturity condition

\[ G(X,t=T) = X_T - X_0 \]

We see that in the derivation of the contract value, \( S/I \) plays the role of the underlying "price" and \( R(S,I,t) \) is the contract value. The analogy with the options market is immediate where \( S \) alone plays the role of the underlying price and where the contract value, for a call, is a function \( C(S,X,t) \).

The solution (4) is the fair value at time t of the expected contract payoff at time T of a contract entered into at time t. The contract is margined so the position-holders will not pay anything when entering the contract at time t but will settle at time T. Hence a finance charge, \( e^{r(T-t)} \), has to be levied over the life, T, of the contract.

The revised pricing equation for the margined ratio contract is

\[
R(S,I,t) = \left[ \frac{S_t}{I_t} \right] e^{-r(T-t)} \left[ 1 - \frac{1}{2}\sigma_r^2 + \frac{1}{2}\sigma_j^2 - \rho \sigma_r \sigma_j \right] XG_x + G_r - r G = 0
\]

This is a more intuitive means of expressing the valuation formula because it encapsulates the relative movement effect between the stock and the index. If the stock beta = 1.0 then movements (returns) in the stock are expected to match movements in the index so that (ignoring dividend effects) the ratio remains unchanged. Such a contract would have zero expected payoff and so it would have zero present value, as confirmed by the formula. We return later to the related issue of expected overperformance or underperformance by the stock.

(ii) This pricing formula (6) is of the form

\[
R(S,I,t) = \left[ \frac{S_t}{I_t} \right] e^{-r(T-t)} \left[ 1 - \frac{1}{2}\sigma_r^2 + \frac{1}{2}\sigma_j^2 - \rho \sigma_r \sigma_j \right] XG_x + G_r - r G = 0
\]

which means its formula is similar to that of a forward. However it has attributes that are not those of forwards so it is not a forward contract. Indeed, it has an
The ratio contract has elements of equity (a stock beta), options (volatility) and a forward contract (but to be settled on the relative performance, not the usual absolute performance of forwards).

(iii) Like options, but unlike forwards or futures contracts, the value of the ratio depends on the volatility (actually the variance) of the underlying securities. In contrast, the value of forwards or futures does not depend on volatility because their payoffs are symmetric. Options do depend on volatility because they have a one-sided (asymmetric) payoff structure. The result for ratios indicates that they too must have some type of asymmetric payoff. This is indeed the case. Suppose the beta of a stock is 2.0. The index:

a. Increases by 10%. The stock increases by 20% and the ratio increases by 9.09% (1.20/1.01 = 1.0909);

b. Decreases by 10%. The stock decreases by 20% and the ratio decreases by 11.11% (0.80/0.90 = 0.8889).

So the ratio contract does have an asymmetric payoff, somewhere between those of an option and a forward or futures.

This example suggests that increasing betas reduce the ratio value for a given index volatility. This is indeed the case, as is seen from the solution:

While this example is extreme, it demonstrates the skewness attribute. In practice, however, the contract will operate on daily settlement (analogous to a futures-type margining) so the contract will be equivalent to a series of daily forward contracts. Such a short life of each forward contract makes the daily effect (within each forward) of the skewness much less dramatic than this extreme example suggests.

(iv) The asymmetry contribution to value can be either positive, zero, or negative, depending on the size of the beta. Suppose the dividend yields are zero or cancel each other:

If $\beta < 1$ then $R(S, I, t) > 0$
If $\beta = 1$ then $R(S, I, t) = 0$
If $\beta > 1$ then $R(S, I, t) < 0$

(iv) The maturity value of the contract is 0, clearly the appropriate payoff upon entering a contract at the price $S_t/I_t$ that will immediately expire with the same price $S_t/I_t$.

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Similarly, the expected return to equilibrium levels. A stock could be experience price increases that will decrease work to pull it back into equilibrium. This, of course, contradicts the expectation of the CAPM. This contract has a small positive value because the dividend yield gap exceeds the asymmetry effect described above. At the moment of writing the ratio, a premium of $5.53 is owed by the long position to the short position. Instead of paying this upfront, like an option premium, it is paid by way of margin (premium erosion), like a futures contract.

\[ [0.04 - 0.03 + (0.17)^2(1-1.24)](0.5) = 0.0015 \]

so the ratio price, \( R(S,L,t) \), is

\[ \$3,609.76[\exp(0.0015)-1] = \$5.53 \]

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OUTPERFORMANCE
The attraction of ratio contracts to funds managers (and investors in general) is that the contract permits an efficient means of capturing expected outperformance by the stock. This excess performance by stock returns over market returns is typically measured by a CAPM model of the form

\[ R_S = \alpha + \beta_R [R_M - R_S] \]

The parameter \( \alpha \) measures overperformance \((\alpha > 0)\), underperformance \((\alpha < 0)\) or fair value performance \((\alpha = 0)\).

The typical method for establishing a derivative contract valuation is based on the no-arbitrage valuation logic. An alternative method could be an equilibrium pricing method (indeed the original Black-Scholes paper contained both methods). However, even in equilibrium there cannot be any arbitrage; otherwise an investor could make a riskless profit and presuming they are non-satiated they would indeed make such a transaction. This, of course, contradicts the assertion of equilibrium.

Not all securities are always earning their equilibrium rate of return. A stock could be underperforming until market corrections work to pull it back into equilibrium. Similarly, an overperforming stock would experience price increases that will decrease the expected return to equilibrium levels. Note that the overperformance or underperformance will only be experienced by current holders of these securities. A seller of an overperforming stock would want the capitalised value of this overperformance in the sale price. Hence the new owners would receive just a fair return. How quickly this adjustment to equilibrium takes place is a function of the demand and supply for that particular stock.

McDonald and Siegal (1984) have solved the option pricing formula when the underlying asset earns a below-equilibrium rate of return. By implication, their argument equally applies to overperformance. Their result is a simple modification to the Black-Scholes formula. It is intuitively understood if we think of underperformance as a “dividend yield” that reduces asset prices, but the shareholder does not collect this dividend. Hence underperformance is like a series of ex-dividend prices where the shareholder misses out on the compensating cash dividends. Similarly, overperformance is like a negative dividend that causes asset prices to rise “ex-dividend.”

We can apply this to the ratio contract by assuming that the stock also experiences this “performance dividend yield”. The mathematics will not be reproduced here because it just mirrors the above derivation with the stock dividend yield supplemented by the “performance” yield. The extended pricing formula is

\[
R(S,I,t) = \left[ \frac{S}{I_t} \right] \times \left[ e^{[d_1 - d_2 + \alpha + \sigma^2(1-\beta_S)(T-t) - \frac{1}{2}\sigma^2]} \right]
\]

(9)

where \( \alpha \) is the performance yield on the stock. If \( \alpha > 0 \) then this is like a negative stock dividend yield so this case corresponds to overperformance. Similarly, \( \alpha < 0 \) behaves like a standard (positive) dividend yield so this case corresponds to underperformance.

This valuation formula is one that would apply to prospective outperformance of stock-specific risk. It can be viewed as a valuation in three steps:

- calculate the dividend yield spread \( d_1 - d_2 \)
- add the asymmetric volatility effect, \( \sigma^2(1-\beta_S) \)
- add the anticipated outperformance yield, \( \alpha \)

But it must be emphasised that this valuation formula cannot apply in equilibrium as then there can be no expected outperformance. The current shareholders may be the beneficiaries of positive value news (such as the company announcing new, positive NPV projects) but this will quickly be priced into the value of actively traded shares.

CONCLUSION
This new contract has generated much discussion in recent months, some of it ill-informed and based on ill-founded notions of the contracts. Ratios are a very elegant and simple contract for capturing stock-specific effects.

The pricing formula presented here is a practical formula that any practitioner in derivatives markets would have no problem in putting into operation.

Numerous people assisted by way of discussion of the issues in this paper. However special acknowledgment is made to the people at ASXD, especially Peter Ho, who played a very active and interested role in the project.

REFERENCES